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ON THE ENERGY FLUX VECTOR FOR BENDING VIBRATIONS OF A PLATE

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An expression for the energy flux vector of plate bending vibrations is obtained in invariant form. The derivation of expressions for the transverse force, bending and twisting moments in an arbitrary orthogonal coordinate system and the derivation of an orthogonality type condition for normal waves being propagated in a thin elastic strip with free edges are considered as applications.

In a number of cases it turns out to be useful to consider the energy flux vector in analyzing the vibrations in systems with distributed parameters. The expressions for the Umov-Poynting vector in electrodynamics and for the energy flux vector in acoustics are well-known. An analogous vector for the bending vibrations of a plate was mentioned only in [1 — 3]. This vector is used in [1] to prove a uniqueness theorem for a two-component acoustic model consisting of an ideal compressible fluid and elastic plates in contact with it. However, the expression for the energy flux in [1] (it was later cited in [2, 3] with a reference to [1]) is erroneous. An exact expression (within the framework of the applicability of the Kirchhoff equation) is found below for the energy flux vector of the bending vibrations of a plate and some applications of the formulas obtained are mentioned.

Let us write the expression for the energy density w of the plate bending vibrations [4]

$$w = \frac{D}{2}\left\{\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right)^2 + 2(1-\sigma)\left[\left(\frac{\partial^2\zeta}{\partial x\,\partial y}\right)^2 - \frac{\partial^2\zeta}{\partial x^2}\frac{\partial^2\zeta}{\partial y^2}\right]\right\} + \frac{\rho}{2}\left(\frac{\partial\zeta}{\partial t}\right)^2 \tag{1}$$

Here $\zeta = \zeta(x, y, t)$ is the plate bending displacement, D is the cylindrical stiffness, ρ is the plate surface density, and σ is the Poisson's ratio. We differentiate (1) with respect to time t and eliminate the second derivative with respect to t using the equation of plate bending vibrations

$$D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2\zeta + \rho\frac{\partial^2\zeta}{\partial t^2} = 0 \tag{2}$$

We consequently obtain after elementary manipulations

$$\begin{aligned}\frac{\partial w}{\partial t} = D\Big\{&2\frac{\partial}{\partial x}\left[\frac{\partial\zeta}{\partial t}\left(\frac{\partial^3\zeta}{\partial x^3} + \frac{\partial^3\zeta}{\partial x\,\partial y^2}\right)\right] + \\ &2\frac{\partial}{\partial y}\left[\frac{\partial\zeta}{\partial t}\left(\frac{\partial^3\zeta}{\partial y^3} + \frac{\partial^3\zeta}{\partial y\,\partial x^2}\right)\right] - \frac{\partial^2}{\partial x^2}\left[\frac{\partial\zeta}{\partial t}\left(\frac{\partial^2\zeta}{\partial x^2} + \sigma\frac{\partial^2\zeta}{\partial y^2}\right)\right] - \\ &\frac{\partial^2}{\partial y^2}\left[\frac{\partial\zeta}{\partial t}\left(\frac{\partial^2\zeta}{\partial y^2} + \sigma\frac{\partial^2\zeta}{\partial x^2}\right)\right] - 2(1-\sigma)\frac{\partial^2}{\partial x\,\partial y}\left(\frac{\partial\zeta}{\partial t}\frac{\partial^2\zeta}{\partial x\,\partial y}\right)\Big\}\end{aligned} \tag{3}$$

Let us introduce the energy flux vector $\mathbf{\Pi}$ by the relationship

$$\partial w/\partial t + \operatorname{div}\mathbf{\Pi} = 0 \tag{4}$$

Equating (3) and (4) and using the two-dimensional Hamilton ∇ and Laplace Δ operators, we obtain

$$\mathbf{\Pi} = -D\left\{2\frac{\partial\zeta}{\partial t}\nabla\Delta\zeta - \sigma\nabla\frac{\partial\zeta}{\partial t}\Delta\zeta - (1-\sigma)\left(\nabla\cdot\frac{\partial\zeta}{\partial t}\nabla\right)\nabla\zeta\right\}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}, \qquad \Delta = (\nabla\cdot\nabla)$$

Here, according to the traditional symbolism [5], it is assumed that the operator ∇ acts on all the quantities in the products after it. The first operator Δ acts on both ζ and $\partial\zeta / \partial t$ in the second and third members on the right. By using the usual methods [5] we can avoid this inconvenience. We hence arrive at the final expression

$$\mathbf{\Pi} = -D\left\{\frac{\partial\zeta}{\partial t}\nabla\Delta\zeta - \sigma\frac{\partial\nabla\zeta}{\partial t}\Delta\zeta - (1-\sigma)\left(\frac{\partial\nabla\zeta}{\partial t}\cdot\nabla\right)\nabla\zeta\right\} \tag{5}$$

We introduce orthogonal curvilinear coordinates q_1, q_2

$$x = x(q_1, q_2), \qquad y = y(q_1, q_2)$$

and we find the component Π_1 of the vector Π along the coordinate q_1 and consider the last term in the right side of (5) in more detail since the passage over to curvilinear coordinates is obvious in the first two members

$$\left(\frac{\partial\nabla\zeta}{\partial t}\cdot\nabla\right)\nabla\zeta = \left(\frac{1}{H_1}\frac{\partial^2\zeta}{\partial q_1\partial t}\frac{1}{H_1}\frac{\partial}{\partial q_1} + \frac{1}{H_2}\frac{\partial^2\zeta}{\partial q_2\partial t}\frac{1}{H_2}\frac{\partial}{\partial q_2}\right)\left(\frac{1}{H_1}\frac{\partial\zeta}{\partial q_1}\mathbf{e}_1 + \frac{1}{H_2}\frac{\partial\zeta}{\partial q_2}\mathbf{e}_2\right)$$

Here H_1, H_2 are Lamé coefficients, $\mathbf{e}_1$, $\mathbf{e}_2$ are unit directions. The inconvenience of the last formula is that differentiation of unit directions is assumed. Using the rule for the differentiation of directions

$$\frac{\partial\mathbf{e}_i}{\partial q_i} = -\frac{1}{H_k}\frac{\partial H_i}{\partial q_k}\mathbf{e}_k, \qquad \frac{\partial\mathbf{e}_i}{\partial q_k} = \frac{1}{H_i}\frac{\partial H_k}{\partial q_i}\mathbf{e}_k$$

$$(i, k = 1, 2; \; i \neq k)$$

we arrive at the following expression for the components of the flux vector Π_1

$$\Pi_1 = F\frac{\partial\zeta}{\partial t} - M\frac{1}{H_1}\frac{\partial^2\zeta}{\partial q_1\partial t} + N\frac{1}{H_2}\frac{\partial^2\zeta}{\partial q_2\partial t} \tag{6}$$

$$F = -D\frac{1}{H_1}\frac{\partial\Delta\zeta}{\partial q_1} \tag{7}$$

$$M = -D\left\{\sigma\Delta\zeta + \frac{(1-\sigma)}{H_1}\left[\frac{\partial}{\partial q_1}\left(\frac{1}{H_1}\frac{\partial\zeta}{\partial q_1}\right) - \frac{1}{H_1H_2}\frac{\partial\zeta}{\partial q_1}\frac{\partial H_1}{\partial q_2}\right]\right\}$$

$$N = D\frac{(1-\sigma)}{H_2}\left[\frac{\partial}{\partial q_2}\left(\frac{1}{H_1}\frac{\partial\zeta}{\partial q_1}\right) - \frac{1}{H_1H_2}\frac{\partial\zeta}{\partial q_2}\frac{\partial H_2}{\partial q_1}\right]$$

We draw an imaginary slit $q_2 = q_2^\circ = \text{const}$ in the plate and we discard the part of the plate for which $q_2 > q_2^\circ$. Then F has the meaning of a transverse force acting on the plate from the discarded part, M is the bending and N the twisting moment. The expressions (7) are considerably more convenient than the traditional formulas [4] in which the Cartesian variables x and y take part, and the passage from which to arbitrary coordinates is related to a direct conversion of the derivatives. In particular, we immediately obtain known expressions for the polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, $q_1 = r$, $q_2 = \varphi$, $H_1 = 1$, $H_2 = r$.

As another application of the result obtained, we consider the derivation of an expression for the vibrational energy flux in an elastic tape. In the interest of brevity, we call this an infinitely long, elastic plate of constant width. The vibrations of such a plate have been studied in [6]. We direct the Oy-axis along the plate; let $x = \pm a$ be the equation of the plate edges.

Equation (2) holds for a plate for $|x| < a$, $y \in (-\infty, +\infty)$. The plate edges are assumed free, which yields the following boundary conditions:

$$F - \frac{\partial N}{\partial y} = -D\left[\frac{\partial^3\zeta}{\partial x^3} + (2-\sigma)\frac{\partial^3\zeta}{\partial x\,\partial y^2}\right] = 0$$
$$M = -D\left(\frac{\partial^2\zeta}{\partial x^2} + \sigma\frac{\partial^2\zeta}{\partial y^2}\right) = 0, \quad x = \pm a \tag{8}$$

Let us imagine a rectangle $ABCD$ (Fig. 1), whose two sides coincide with the plate edges, cut out of the plate, and let us calculate the energy flux Π through its outline. We use the Green's formula

$$\iint\limits_{ABCD} div\Pi\,dx\,dy = \int_A^B \Pi_x\,dy - \int_B^C \Pi_y\,dx + \int_C^D \Pi_x\,dy - \int_D^A \Pi_y\,dx$$

By virtue of (5) the component Π_x can be written as follows:

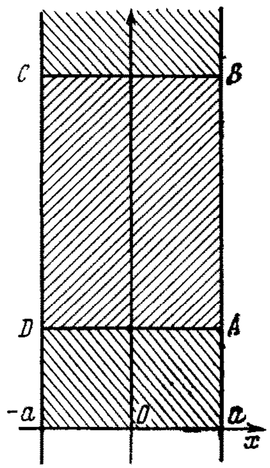

Fig. 1

$$\Pi_x = -D\left[\frac{\partial\zeta}{\partial t}\left(\frac{\partial^3\zeta}{\partial x^3} + \frac{\partial^3\zeta}{\partial x\,\partial y^2}\right) - (1-\sigma)\frac{\partial^2\zeta}{\partial y\,\partial t}\frac{\partial^2\zeta}{\partial x\,\partial y} - \frac{\partial^2\zeta}{\partial x\,\partial t}\left(\frac{\partial^2\zeta}{\partial x^2} + \sigma\frac{\partial^2\zeta}{\partial y^2}\right)\right] \tag{9}$$

The expression for Π_y is obtained from this latter by a circular permutation $x \to y \to x$. In evaluating the integrals of Π_x, we integrate by parts in the second member, and we then use the boundary conditions (8). The integrands hence vanish, consequently

$$\int_A^B \Pi_x\,dy = D(1-\sigma)\frac{\partial\zeta}{\partial t}\frac{\partial^2\zeta}{\partial x\,\partial y}\Big|_A^B$$

We also integrate by parts in the second term when evaluating the integrals of Π_y, and obtain

$$\iint\limits_{ABCD} \operatorname{div}\mathbf{\Pi}\,dx\,dy =$$
$$D\int_B^C \left\{\frac{\partial^2\zeta}{\partial y\,\partial t}\left(\frac{\partial^2\zeta}{\partial y^2} + \sigma\frac{\partial^2\zeta}{\partial x^2}\right) - \frac{\partial\zeta}{\partial t}\left[\frac{\partial^3\zeta}{\partial y^3} + (2-\sigma)\frac{\partial^3\zeta}{\partial y\,\partial x^2}\right]\right\}dx +$$
$$D\int_D^A \left\{\frac{\partial^2\zeta}{\partial y\,\partial t}\left(\frac{\partial^2\zeta}{\partial y^2} + \sigma\frac{\partial^2\zeta}{\partial x^2}\right) - \frac{\partial\zeta}{\partial t}\left[\frac{\partial^3\zeta}{\partial y^3} + (2-\sigma)\frac{\partial^3\zeta}{\partial y\,\partial x^2}\right]\right\}dx +$$
$$2D(1-\sigma)\left(\frac{\partial\zeta}{\partial t}\frac{\partial^2\zeta}{\partial x\,\partial y}\Big|_B^C + \frac{\partial\zeta}{\partial t}\frac{\partial^2\zeta}{\partial x\,\partial y}\Big|_D^A\right)$$

In the absence of external field sources $\operatorname{div}\Pi = 0$, therefore, in this case the expression

$$\Pi = 2D(1-\sigma)\frac{\partial\zeta}{\partial t}\frac{\partial^2\zeta}{\partial x\,\partial y}\Big|_{x=-a}^{x=a} +$$
$$D\int_{-a}^{a} \left\{\frac{\partial^2\zeta}{\partial y\,\partial t}\left(\frac{\partial^2\zeta}{\partial y^2} + \sigma\frac{\partial^2\zeta}{\partial x^2}\right) - \frac{\partial\zeta}{\partial t}\left[\frac{\partial^3\zeta}{\partial y^3} + (2-\sigma)\frac{\partial^3\zeta}{\partial y\,\partial x^2}\right]\right\}dx \tag{10}$$

is independent of the selection of y. This expression has the meanimg of bending vibrations energy transferred per unit time through an arbitrary normal section of elastic tape in the direction of increasing coordinate y. Attention should be directed to the presence of the term outside the integral in (10).

We now turn to an examination of stationary processes. Let us assume that the time dependence is given by the factor $\exp(-i\omega t)$ and let us agree to always omit it. As usual, we shall take the real part of the complex quantity $\zeta(x, y) \exp(-i\omega t)$ as the displacement in the plate. In the general case we have for the energy flux vector (5) after taking the average with respect to time

$$\langle \Pi \rangle = -\frac{D\omega}{2} \operatorname{Im} \{\zeta^* \nabla \Delta \zeta - \sigma \nabla \zeta^* \Delta \zeta - (1-\sigma)(\nabla \zeta^* \cdot \nabla) \nabla \zeta\}$$

where the asterisk denotes the complex conjugate.

Correspondingly, we obtain for the energy flux in an elastic tape (10)

$$\langle \Pi \rangle = 2D\omega(1-\sigma) \operatorname{Im} \zeta^* \frac{\partial^2 \zeta}{\partial x \, \partial y} \Big|_{x=-a}^{x=a} +$$

$$\frac{D\omega}{2} \operatorname{Im} \int_{-a}^{a} \left\{ \frac{\partial \zeta^*}{\partial y} \left(\frac{\partial^2 \zeta}{\partial y^2} + \sigma \frac{\partial^2 \zeta}{\partial x^2} \right) - \zeta^* \left[\frac{\partial^3 \zeta}{\partial y^3} + (2-\sigma) \frac{\partial^3 \zeta}{\partial y \, \partial x^2} \right] \right\} dx$$

Let us introduce two normal waves [6]

$$\zeta_1 = u_1(x) e^{i\mu_1 y}, \quad \zeta_2 = u_2(x) e^{i\mu_2 y}$$

$$(\operatorname{Im} u_{1,2} = 0, \quad \operatorname{Im} \mu_1, \mu_2 = 0)$$

Which are propagated in the elastic tape with the wave numbers μ_1 and μ_2 ($\mu_1 \neq \mu_2$). We evidently have for the energy flux $\langle \Pi \rangle$ of their superpositions $\zeta = \zeta_1 + \zeta_2$

$$\langle \Pi \rangle = \langle \Pi_1 \rangle + \langle \Pi_2 \rangle + \langle \Pi_{12} \rangle \tag{11}$$

Here $\langle \Pi_1 \rangle$, $\langle \Pi_2 \rangle$ are the energy fluxes transferred separately by the waves ζ_1 and ζ_2. After certain calculations we obtain for the energy flux $\langle \Pi_{12} \rangle$ of their interaction

$$\langle \Pi_{12} \rangle = \frac{D\omega}{2} (\mu_1 + \mu_2) \cos[(\mu_1 - \mu_2) y] \Big\{ \sigma (u_2 u_1' + u_1 u_2') \Big|_{-a}^{a} + \int_{-a}^{a} [(\mu_1^2 + \mu_2^2) u_1 u_2 + 2u_1' u_2'] \, dx \Big\} \tag{12}$$

The quantities $\langle \Pi \rangle$, $\langle \Pi_1 \rangle$, $\langle \Pi_2 \rangle$ are independent of y, whereupon $\langle \Pi_{12} \rangle$ should also be independent of y by virtue of (11). Hence, the expression in the braces in (12) should equal zero. We consequently arrive at the following condition of orthogonality type [7] for normal waves in an elastic tape

$$\sigma (u_1 u_2' + u_2 u_1') \Big|_{-a}^{a} + \int_{-a}^{a} [(\mu_1^2 + \mu_2^2) u_1 u_2 + 2u_1' u_2'] \, dx = 0$$

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EXISTENCE OF A GLOBAL LIAPUNOV FUNCTIONAL FOR CERTAIN CLASSES OF NONLINEAR DISTRIBUTED SYSTEMS

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The existence of a global Liapunov functional for nonlinear evolutionary equations in a Hilbert space is investigated as a continuation of paper [1]. The results obtained represent a generalization of the results of the theory of absolute stability [2, 3], for the systems with infinite dimensional phase space, and are used for investigation of the nonlocal stability and instability of nonlinear distributed systems. The conditions of existence of the global Liapunov functional obtained are illustrated by an example of a nonlinear parabolic system defined in the interval [0, 1]. The concept of a Liapunov functional was first introduced and used with success in [4].

1. Evolutionary equations in a Hilbert space. Class N of nonlinear operators. Let H. V and U be the Hilbert spaces [5] over a field of real numbers with scalar products $\langle , \rangle_H$, $\langle , \rangle_V$ and $\langle , \rangle_U$, and zero elements θ_H, θ_V and θ_U, respectively. We denote by H^* and V^* the Hilbert spaces conjugate to H and V [5], assume that $V \subset H = H^* \subset V^*$, that the space V is dense in H and, that the imbedding $V \to H$ is continuous. Let A be a continuous nonlinear operator $V \to V^*$ closed in the space H. Further, let B be a linear bounded operator $U \to V^*$ and $\Phi(\cdot)$ a nonlinear (generally speaking) operator $H \times R^1 \to U$, where R^1 denotes the real axis.

We consider the following nonlinear evolutionary equation [6]:

$$\frac{d}{dt} x(t) = Ax(t) + B\Phi(x(t), t) \tag{1.1}$$

By the generalized solution of (1.1) in the interval (τ, T) we understand the function $x(t) \in W(\tau, T; V)$ satisfying the equation

$$\int_\tau^T \left[\left\langle x(t), \frac{d\xi(t)}{dt}\right\rangle + \langle Ax(t), \xi(t)\rangle + \langle B\Phi(x(t), t), \xi(t)\rangle\right] dt = 0 \tag{1.2}$$